



8.8 The vector space of linear maps

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Introduction

- In this topic, we will
 - Define scalar multiplication and addition of linear maps
 - Consider the consequences for matrices
 - Prove that the vector space of all such linear maps is itself a vector space
 - The vectors in this vector space are the linear maps
 - Consider a definition for the norm of such a vector





Multiplying a linear map by a scalar

- In order to claim that linear maps are a vector space, we need a definition of:
 - Scalar multiplication of a linear map
 - The sum of two linear maps
- We will see how these definitions apply to matrices
- We will then prove that using these definitions, the collection of all linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$ forms a vector space
 - Important: each pair of vector spaces $\mathcal{U}$ and $\mathcal{V}$ defines a different vector space of linear maps





Multiplying a linear map by a scalar

- Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, and a scalar $\alpha \in \mathbf{F}$, we will define the linear map $(\alpha A): \mathcal{U} \rightarrow \mathcal{V}$ where

$$(\alpha A)\mathbf{u} = \alpha (A\mathbf{u})$$

- That is, the map αA applied to $\mathbf{u}$ has the same effect of
 - Applying A to a vector $\mathbf{u}$
 - Multiplying that result by α





Scalar multiplication of a matrix

- Suppose $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ and $\alpha \in \mathbf{F}$
 - Question: What is αA ?

- Recall that if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, then

$$A\mathbf{u} = u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n$$

- Thus, we have that that if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, then

$$\begin{aligned}(\alpha A)\mathbf{u} &= \alpha(A\mathbf{u}) \\&= \alpha(u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n) \\&= \alpha(u_1\mathbf{a}_1) + \cdots + \alpha(u_n\mathbf{a}_n) \\&= u_1(\alpha\mathbf{a}_1) + \cdots + u_n(\alpha\mathbf{a}_n) \\&= (\alpha\mathbf{a}_1 \cdots \alpha\mathbf{a}_n)\mathbf{u}\end{aligned}$$

- Thus, αA is every entry of A multiplied by α





Example

- For example, $A: \mathbf{R}^6 \rightarrow \mathbf{R}^4$ and $-1.5 \in \mathbf{R}$

$$A = \begin{pmatrix} 2.3 & -1.2 & -0.5 & 2.5 & 3.1 & -1.3 \\ 2.1 & 4.5 & 3.2 & -1.3 & -2.4 & 0.9 \\ 4.1 & 1.3 & 1.2 & -1.8 & 1.3 & 0.4 \\ -1.3 & -1.5 & -1.8 & 1.0 & -1.3 & -2.4 \end{pmatrix}$$

- Question: What is $-1.5 \cdot A$?

$$-1.5 \cdot A = \begin{pmatrix} -3.45 & 1.8 & 0.75 & -3.75 & -4.65 & 1.3 \\ -3.15 & -6.75 & -4.8 & 1.95 & 3.6 & -1.35 \\ -6.15 & -1.95 & -1.8 & 2.7 & -1.95 & -0.6 \\ 1.95 & 2.25 & 2.7 & -1.5 & 1.95 & 3.6 \end{pmatrix}$$





Adding two linear maps

- Similarly, given linear maps $A, B: \mathcal{U} \rightarrow \mathcal{V}$, we will define the linear map $(A + B): \mathcal{U} \rightarrow \mathcal{V}$ where

$$(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$$

- That is, the map $A + B$ is that map that has the same effect of
 - Applying both A and B to a vector $\mathbf{u}$, separately
 - Adding the resultant vectors





Addition of two matrices

- Suppose $A, B: \mathbf{F}^n \rightarrow \mathbf{F}^m$
 - Question: What is $A + B$?

- Recall that if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ and $B = (\mathbf{b}_1 \cdots \mathbf{b}_n)$, then

$$A\mathbf{u} = u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n \text{ and } B\mathbf{u} = u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n$$

- Thus, we have that

$$\begin{aligned}(A + B)\mathbf{u} &= A\mathbf{u} + B\mathbf{u} \\ &= (u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n) + (u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n) \\ &= u_1(\mathbf{a}_1 + \mathbf{b}_1) + \cdots + u_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= (\mathbf{a}_1 + \mathbf{b}_1 \quad \cdots \quad \mathbf{a}_n + \mathbf{b}_n) \mathbf{u}\end{aligned}$$

- Thus, $A + B$ sums the corresponding entries of A and B





Example

- For example, $A, B: \mathbf{R}^4 \rightarrow \mathbf{R}^3$

$$A = \begin{pmatrix} 2.5 & 8.9 & -7.1 & 0.3 \\ 6.4 & -5.7 & 9.0 & -1.8 \\ -0.8 & 1.9 & 3.5 & -2.4 \end{pmatrix} \quad B = \begin{pmatrix} 3.9 & 2.5 & 3.8 & -1.6 \\ 2.8 & 8.7 & -1.4 & -5.8 \\ -2.0 & -4.6 & 3.7 & -1.9 \end{pmatrix}$$

- Question: What is $A + B$?

$$A + B = \begin{pmatrix} 6.4 & 11.4 & -3.3 & -1.3 \\ 9.2 & 3.0 & 7.6 & -7.6 \\ -2.8 & -2.7 & 7.2 & -4.3 \end{pmatrix}$$





Example

- Why did we not just define αA and $A + B$ for matrices?
 - First, not all linear maps are matrices
 - Second, we derived the formulas for scalar multiplication and matrix addition based on the general definitions of linear maps
 - Finally, we will later define composition of linear maps, and from the definition, we will deduce matrix-matrix multiplication
 - Hint, we will not be multiplying two $m \times n$ matrices elementwise





Linear maps are a vector space

- We will show that the collection of all linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$ forms a vector space over $\mathbf{F}$
- For many of the proofs, we will show that one linear comb applied to an arbitrary vector $\mathbf{u}$ equals a different expression for an arbitrary vector $\mathbf{u}$
 - We said two maps A and B are equal if $A\mathbf{u} = B\mathbf{u}$ for all $\mathbf{u}$
- For each step, we will either
 - Apply one of the definitions of a linear map
$$(\alpha A) \mathbf{u} = \alpha (A\mathbf{u}) \quad \text{or} \quad (A + B) \mathbf{u} = A\mathbf{u} + B\mathbf{u}$$
or a property of vector spaces





Axiom 1

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, then αA is a linear map.

Proof:

$$\begin{aligned}(\alpha A)(\beta \mathbf{u}_1 + \gamma \mathbf{u}_2) &= \alpha A(\beta \mathbf{u}_1 + \gamma \mathbf{u}_2) \\&= \alpha (\beta A\mathbf{u}_1 + \gamma A\mathbf{u}_2) \\&= \alpha (\beta A\mathbf{u}_1) + \alpha (\gamma A\mathbf{u}_2) \\&= \beta (\alpha A\mathbf{u}_1) + \gamma (\alpha A\mathbf{u}_2) \\&= \beta (\alpha A)\mathbf{u}_1 + \gamma (\alpha A)\mathbf{u}_2 \quad \blacksquare\end{aligned}$$





Axiom 2

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, then $1A = A$.

Proof:

$$\begin{aligned}(1 \cdot A)\mathbf{u} &= 1 \cdot A\mathbf{u} \\ &= A\mathbf{u} \quad \blacksquare\end{aligned}$$





Axiom 3

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, then $(\alpha\beta)A = \alpha(\beta A)$.

Proof:

$$\begin{aligned} ((\alpha\beta)A) \mathbf{u} &= (\alpha\beta)(A\mathbf{u}) \\ &= \alpha(\beta A\mathbf{u}) \\ &= \alpha((\beta A)\mathbf{u}) \\ &= (\alpha(\beta A)) \mathbf{u} \quad \blacksquare \end{aligned}$$





Axiom 4

Theorem

If $A, B: \mathcal{U} \rightarrow \mathcal{V}$ are linear maps, then $A + B$ is a linear map.

Proof:

$$\begin{aligned}(A + B)(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) &= A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) + B(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) \\&= \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2 + \alpha B\mathbf{u}_1 + \beta B\mathbf{u}_2 \\&= \alpha (A\mathbf{u}_1 + B\mathbf{u}_1) + \beta (A\mathbf{u}_2 + B\mathbf{u}_2) \\&= \alpha (A + B)\mathbf{u}_1 + \beta (A + B)\mathbf{u}_2 \blacksquare\end{aligned}$$





Axiom 5

Theorem

If $A, B: \mathcal{U} \rightarrow \mathcal{V}$ are linear maps, then $A + B = B + A$.

Proof:

$$\begin{aligned}(A + B) \mathbf{u} &= A\mathbf{u} + B\mathbf{u} \\ &= B\mathbf{u} + A\mathbf{u} \\ &= (B + A) \mathbf{u} \quad \blacksquare\end{aligned}$$





Axiom 6

Theorem

If $A, B, C: \mathcal{U} \rightarrow \mathcal{V}$ are linear maps,
then $(A + B) + C = A + (B + C)$.

Proof:

$$\begin{aligned} ((A + B) + C) \mathbf{u} &= (A + B)\mathbf{u} + C\mathbf{u} \\ &= (A\mathbf{u} + B\mathbf{u}) + C\mathbf{u} \\ &= A\mathbf{u} + (B\mathbf{u} + C\mathbf{u}) \\ &= A\mathbf{u} + (B + C)\mathbf{u} \\ &= (A + (B + C))\mathbf{u} \quad \blacksquare \end{aligned}$$





Axiom 7

Theorem

If $A, B: \mathcal{U} \rightarrow \mathcal{V}$ are linear maps, $\alpha(A + B) = \alpha A + \alpha B$.

Proof:

$$\begin{aligned}(\alpha(A + B)) \mathbf{u} &= \alpha((A + B) \mathbf{u}) \\&= \alpha(A\mathbf{u} + B\mathbf{u}) \\&= \alpha(A\mathbf{u}) + \alpha(B\mathbf{u}) \\&= (\alpha A)\mathbf{u} + (\alpha B)\mathbf{u} \\&= (\alpha A + \alpha B)\mathbf{u} \quad \blacksquare\end{aligned}$$





Axiom 8

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, $(\alpha + \beta)A = \alpha A + \beta A$.

Proof:

$$\begin{aligned} ((\alpha + \beta)A) \mathbf{u} &= (\alpha + \beta)(A \mathbf{u}) \\ &= \alpha(A \mathbf{u}) + \beta(A \mathbf{u}) \\ &= (\alpha A)\mathbf{u} + (\beta A)\mathbf{u} \\ &= (\alpha A + \beta A)\mathbf{u} \quad \blacksquare \end{aligned}$$





Axiom 9

Theorem

The zero map $O: \mathcal{U} \rightarrow \mathcal{V}$ where $O\mathbf{u} = \mathbf{0}_{\mathcal{V}}$ satisfies $A + O = A$ for all linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$.

Proof:

$$\begin{aligned}(O + A)\mathbf{u} &= O\mathbf{u} + A\mathbf{u} \\ &= \mathbf{0}_{\mathcal{V}} + A\mathbf{u} \\ &= A\mathbf{u} \quad \blacksquare\end{aligned}$$





Axiom 10

Theorem

For any $A: \mathcal{U} \rightarrow \mathcal{V}$, define $(-A)\mathbf{u} = -(A\mathbf{u})$, in which case,
 $A + (-A) = O$.

Proof:

$$\begin{aligned}(A + (-A)) \mathbf{u} &= A\mathbf{u} + (-A)\mathbf{u} \\ &= A\mathbf{u} + -(A\mathbf{u}) \\ &= \mathbf{0}_{\mathcal{V}} \quad \blacksquare\end{aligned}$$





Example

- Thus, we showed that the collection of all linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$ forms a vector space over $\mathbf{F}$
 - The individual maps are the *vectors* in this vector space
 - This vector space is written as $\mathcal{L}(\mathcal{U}, \mathcal{V})$
 - For example, $\mathcal{L}(\mathbf{R}^4, \mathbf{R}^3)$ is the vector space of all linear maps that map $\mathbf{R}^4$ to $\mathbf{R}^3$
 - It is the collection of all 3×4 real-valued matrices





Example

- Recall our theorem: if $A \in \mathcal{L}(\mathbf{R}^4, \mathbf{R}^3)$, then $-A = (-1)A$, so if

$$A = \begin{pmatrix} 2.5 & 8.9 & -7.1 & 0.3 \\ 6.4 & -5.7 & 9.0 & -1.8 \\ -0.8 & 1.9 & 3.5 & -2.4 \end{pmatrix}$$

then automatically,

$$-A = (-1)A = \begin{pmatrix} -2.5 & -8.9 & 7.1 & -0.3 \\ -6.4 & 5.7 & -9.0 & 1.8 \\ 0.8 & -1.9 & -3.5 & 2.4 \end{pmatrix}$$

- Also, the zero element can be found by multiplying any matrix by zero, so

$$0 \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = O_{3,4}$$





Vector space of linear operators

- If the domain and codomain are equal, the vector space of all linear operators $A: \mathcal{U} \rightarrow \mathcal{U}$ forms a special vector space
 - This vector space is written as $\mathcal{L}(\mathcal{U})$
 - For example, $\mathcal{L}(\mathbf{R}^4)$ is the vector space of all linear maps that map $\mathbf{R}^4$ to itself
 - It is the collection of all 4×4 real-valued square matrices





Example with derivatives

- The 2nd derivative, the derivative, and the identity operator are linear operators on polynomials, so
 - We may write that $\frac{d^2}{dt^2}, \frac{d}{dt}, I \in \mathcal{L}(\mathbf{R}[t])$
 - You may recall $\mathbf{R}[t]$ is the vector space of all polynomials with real coefficients of a real variable
- Thus, a linear combination of linear operators is a linear operator:

$$\frac{d^2}{dt^2} + 3\frac{d}{dt} + 7I$$

- If $p(t) = 5t^2 + 2t - 1$, then

$$\begin{aligned} \left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 7I \right) p(t) &= \frac{d^2}{dt^2} p(t) + 3\frac{d}{dt} p(t) + 7Ip(t) \\ &= 10 + 3(10t + 2) + 7(5t^2 + 2t - 1) = 35t^2 + 44t + 9 \end{aligned}$$





Norms on matrices

- Can we define a norm on a matrix?
 - How about the square root of the sum of the squares of the entries?
 - Unfortunately, this definition has no practical use, consequently, very few people use it
- How about the following?
 - The 2-norm of a linear map A is the maximum that map stretches the 2-norm of a vector in the domain?

$$\|A\|_2 = \max_{\substack{\mathbf{u} \in \mathcal{U} \\ \mathbf{u} \neq \mathbf{0}}} \left\{ \frac{\|A\mathbf{u}\|_2}{\|\mathbf{u}\|_2} \right\}$$

- Under this definition, $\mathcal{L}(\mathcal{U}, \mathcal{V})$ is a normed vector space





Inner product on matrices?

- One may think you could define the inner product on matrices just like we defined the inner product on vectors:
 - It is the sum of the element-wise multiplications of the two matrices...
 - Unfortunately, this inner product has no practical applications, so while one *could* use this definition, no one does





Summary

- Following this topic, you now
 - Know the collection of all linear maps from one vector space to another forms a vector space
 - Know the definition of scalar multiplication of linear maps, and the addition of linear maps
 - Have reviewed proofs that this is a vector space
 - Recall, the vectors in this vector space are the linear maps!
 - Are aware of the norm of a linear map





References

- [1] https://en.wikipedia.org/wiki/Linear_map





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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